

ON THE STRUCTURE OF \mathcal{N}_p -SPACES IN THE BALLBINGYANG HU[†], LE HAI KHOI[†] AND TRIEU LE

ABSTRACT. We study the structure of \mathcal{N}_p -spaces in the ball. In particular, we show that any such space is Moebius-invariant and for $0 < p \leq n$, all \mathcal{N}_p -spaces are different. Our results will be of important uses in the study of operator theory on \mathcal{N}_p -spaces.

1. INTRODUCTION

1.1. Basic notation and definitions. Throughout the paper, n is a positive integer. Let \mathbb{B} be the open unit ball in \mathbb{C}^n with \mathbb{S} as its boundary. The space $\mathcal{O}(\mathbb{B})$ consists of all holomorphic functions in \mathbb{B} with the compact-open topology. Banach and Hilbert spaces of holomorphic functions on \mathbb{B} have attracted a great attention of researchers, including function theorists and operator theorists. The books [4, 13, 14] are excellent sources for information on these spaces, which include Hardy, Bloch, Bergman and Bergman-type spaces, among others.

A motivation of our work comes from the class of Q_p -spaces on the open unit disk \mathbb{D} on the complex plane. The background on these spaces can be found in the book [12]. In short, for $p > 0$, the Q_p -space consists of functions in $\mathcal{O}(\mathbb{D})$ such that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) < \infty.$$

Here $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ is the automorphism of \mathbb{D} that changes 0 and a ; and dA is the Lebesgue area measure on the plane, normalized so that $A(\mathbb{D}) = 1$. It is known that Q_p -spaces coincide with the classical Bloch space \mathcal{B} for $p \in (1, \infty)$; Q_1 is equal to BMOA, the space of holomorphic functions on \mathbb{D} with bounded mean oscillation; and for $p \in (0, 1)$, the Q_p -spaces are all different.

If, in the definition of the Q_p -space, $f'(z)$ is replaced by $f(z)$, then we have the so-called \mathcal{N}_p -space in the unit disk \mathbb{D} , which was first introduced and studied in [6] and then in [11]. The space \mathcal{N}_p consists of functions in $\mathcal{O}(\mathbb{D})$ for which

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) < \infty.$$

Some properties of \mathcal{N}_p -spaces are: for $p > 1$, the \mathcal{N}_p -space coincides with the Bergman-type space A^{-1} consisting of holomorphic functions on the disk for which $\sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2) < \infty$; and for $p \in (0, 1]$, the \mathcal{N}_p -spaces are all different.

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Several results on different properties of composition operators as well as weighted composition operators acting on \mathcal{N}_p -spaces, or from \mathcal{N}_p -spaces into Bergman-type spaces have been obtained in [6, 11].

1.2. \mathcal{N}_p -spaces in the ball. With the aim to generalize \mathcal{N}_p -spaces to higher dimensions, namely in the unit ball \mathbb{B} of \mathbb{C}^n , in [2], for $p > 0$, the first two authors introduced the \mathcal{N}_p -space of \mathbb{B} as follows:

$$\begin{aligned} \mathcal{N}_p &= \mathcal{N}_p(\mathbb{B}) \\ &= \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_p = \sup_{a \in \mathbb{B}} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \right)^{1/2} < \infty \right\}, \end{aligned}$$

where $dV(z)$ is the normalized volume measure over \mathbb{B} and $\Phi_a \in \text{Aut}(\mathbb{B})$ is the involutive automorphism that interchanges 0 and $a \in \mathbb{B}$ (see, e.g. [7, Chapter 2]).

For $p > 0$, we denote by $A^{-p}(\mathbb{B})$ a Bergman-type space consisting of functions $f \in \mathcal{O}(\mathbb{B})$ for which $|f|_p = \sup_{z \in \mathbb{B}} |f(z)|(1 - |z|^2)^p < \infty$. In [2], several basic properties of \mathcal{N}_p -spaces have been proved, in connection with the Bergman-type spaces A^{-q} . In particular, an embedding theorem for \mathcal{N}_p -spaces and A^{-q} was obtained, together with other useful properties.

Theorem 1.1. [2] *The following statements hold:*

- (a) For $p > q > 0$, we have $H^\infty \hookrightarrow \mathcal{N}_q \hookrightarrow \mathcal{N}_p \hookrightarrow A^{-\frac{n+1}{2}}$.
- (b) For $p > 0$, if $p > 2k - 1, k \in (0, \frac{n+1}{2}]$, then $A^{-k} \hookrightarrow \mathcal{N}_p$. In particular, when $p > n$, $\mathcal{N}_p = A^{-\frac{n+1}{2}}$.
- (c) \mathcal{N}_p is a functional Banach space with the norm $\|\cdot\|_p$, and moreover, its norm topology is stronger than the compact-open topology.
- (d) For $0 < p < \infty$, $\mathcal{B} \hookrightarrow \mathcal{N}_p$, where \mathcal{B} is the Bloch space in \mathbb{B} .

For a holomorphic self-mapping φ of \mathbb{B} and a holomorphic function $u : \mathbb{B} \rightarrow \mathbb{C}$, the linear operator $W_{u,\varphi} : \mathcal{O}(\mathbb{B}) \rightarrow \mathcal{O}(\mathbb{B})$ is called a *weighted composition operator* with symbols u and φ if

$$W_{u,\varphi}(f)(z) = u(z) \cdot (f \circ \varphi(z)), f \in \mathcal{O}(\mathbb{B}), z \in \mathbb{B}.$$

Observe that if u is identically 1, then $W_{u,\varphi} = C_\varphi$ is the *composition operator*, and if φ is the identity, then $W_{u,\varphi} = M_u$ is the *multiplication operator*. The books [1, 9] are excellent sources on composition operators on analytic function spaces.

Considering weighted composition operators between \mathcal{N}_p and Bergman-type spaces A^{-q} , the properties above allowed us to prove criteria for boundedness and compactness of these operators [2, Theorems 3.2 and 3.4]. Furthermore, in [3] the compact differences of weighted composition operators $W_{u,\varphi}$ acting from \mathcal{N}_p -space to the space A^{-q} were considered. Different properties stated in Theorem 1.1 were used.

The structure of this paper is as follows: in Section 2 we show that the space of multipliers of the \mathcal{N}_p -space is precisely the space H^∞ of bounded holomorphic functions on \mathbb{B} . We also show that \mathcal{N}_p -space is a Moebius invariant, which is derived from the isometry property of some class of weighted composition operators. Section 3 deals with the closure of polynomials in \mathcal{N}_p -spaces. Here we introduce the little space \mathcal{N}_p^0 of \mathcal{N}_p which plays an important role in the proof of the density of polynomials in \mathcal{N}_p . In Section 4 we establish criteria for a function to be in \mathcal{N}_p (respectively, in \mathcal{N}_p^0) via p -Carleson measure (respectively, vanishing p -Carleson

measure). Section 5 is devoted to the result that for small values of p (that is $0 < p \leq n$), all \mathcal{N}_p -spaces are different, and so the relationship between \mathcal{N}_p -spaces is given completely. Here functions in the Hadamard gap class are used.

We remark that although the \mathcal{N}_p -spaces are closely related to the \mathcal{Q}_p -spaces, our approach and techniques used in the present paper are different from those for \mathcal{Q}_p -spaces. Moreover, they have their own interests.

Throughout this paper, $d\sigma$ denotes the normalized surface measure on the boundary \mathbb{S} of \mathbb{B} . For $a, b \in \mathbb{R}$, $a \lesssim b$ ($a \gtrsim b$, respectively) means there exists a positive number C , which is independent of a and b , such that $a \leq Cb$ ($a \geq Cb$, respectively). If both $a \lesssim b$ and $a \gtrsim b$ hold, we write $a \simeq b$.

2. MULTIPLIERS AND ISOMETRIC WEIGHTED COMPOSITION OPERATORS

2.1. Multipliers and \mathcal{M} -invariance of \mathcal{N}_p -spaces. We first describe the space $\text{Mult}(\mathcal{N}_p)$ of multipliers of \mathcal{N}_p . Recall that a function $u : \mathbb{B} \rightarrow \mathbb{C}$ is a *multiplier* of \mathcal{N}_p if uf belongs to \mathcal{N}_p for all f in \mathcal{N}_p . An application of the closed graph theorem shows that for any $u \in \text{Mult}(\mathcal{N}_p)$, the multiplication operator M_u is bounded on \mathcal{N}_p .

Proposition 2.1. *For any $p > 0$, we have $\text{Mult}(\mathcal{N}_p) = H^\infty$. Furthermore, for any $u \in H^\infty$, $\|M_u\| = \|u\|_\infty$.*

Proof. For $u \in H^\infty$ and $f \in \mathcal{N}_p$, the function uf belongs to $\mathcal{O}(\mathbb{B})$ and it follows immediately from the definition of the norm in \mathcal{N}_p that

$$\|uf\|_p \leq \|u\|_\infty \|f\|_p.$$

This shows that $H^\infty \subset \text{Mult}(\mathcal{N}_p)$ and $\|M_u\| \leq \|u\|_\infty$. Now suppose that u is an element in $\text{Mult}(\mathcal{N}_p)$. For any integer $m \geq 1$, we have

$$\|u^m\|_p = \|M_u^m 1\|_p \leq \|M_u\|^m \|1\|_p.$$

Combining with Theorem 1.1, we obtain a positive constant $C > 0$ independent of u, m and z such that for any $z \in \mathbb{B}$

$$|u^m(z)| \leq C(1 - |z|^2)^{-(n+1)/2} \|u^m\|_p \leq C(1 - |z|^2)^{-(n+1)/2} \|1\|_p \|M_u\|^m.$$

Consequently,

$$|u(z)| \leq \left(C(1 - |z|^2)^{-(n+1)/2} \|1\|_p \right)^{1/m} \|M_u\|.$$

Letting $m \rightarrow \infty$, we conclude that $|u(z)| \leq \|M_u\|$ for all $z \in \mathbb{B}$. Therefore, u belongs to H^∞ and $\|u\|_\infty \leq \|M_u\|$. This completes the proof of the theorem. \square

By using weighted composition operators with particular symbols, we obtain an alternate description of the norm in \mathcal{N}_p .

For each $w \in \mathbb{B}$, set

$$(2.1) \quad k_w(z) = \left(\frac{1 - |w|^2}{(1 - \langle z, w \rangle)^2} \right)^{\frac{n+1}{2}}, \quad z \in \mathbb{B}.$$

Such k_w is a normalized reproducing kernel function in the Bergman space A^2 . By [2, Lemma 3.1], we have $k_w \in \mathcal{N}_p$ and $\sup_{w \in \mathbb{B}} \|k_w\|_p \leq 1$. Note that for all $w \in \mathbb{B}$, we have $k_w(w) = (1 - |w|^2)^{-(n+1)/2}$.

Furthermore, for any $\Phi \in \text{Aut}(\mathbb{B})$, by [7, Theorem 2.2.5], there exists a unitary operator U such that $\Phi = U\Phi_a$, where $a = \Phi^{-1}(0)$. This shows that $|\Phi(z)| = |\Phi_a(z)|$ for all $z \in \mathbb{B}$. Consequently, we obtain

$$\begin{aligned} \|f\|_p &= \sup_{a \in \mathbb{B}} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \right)^{1/2} \\ &= \sup_{\Phi \in \text{Aut}(\mathbb{B})} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi(z)|^2)^p dV(z) \right)^{1/2} \\ &= \sup_{\Phi \in \text{Aut}(\mathbb{B})} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi^{-1}(z)|^2)^p dV(z) \right)^{1/2}. \end{aligned}$$

For $\Phi \in \text{Aut}(\mathbb{B})$, let $a = \Phi^{-1}(0)$. Denote by W_Φ the weighted composition operator $W_{k_a, \Phi}$. By the change of variables $z = \Phi(w)$ (see [7, Theorem 2.2.6]), we obtain

$$\begin{aligned} &\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi^{-1}(z)|^2)^p dV(z) \\ &= \int_{\mathbb{B}} |f(\Phi(w))|^2 (1 - |w|^2)^p \left(\frac{1 - |a|^2}{|1 - \langle w, a \rangle|^2} \right)^{n+1} dV(w) \\ &= \int_{\mathbb{B}} |f(\Phi(w))|^2 |k_a(w)|^2 (1 - |w|^2)^p dV(w) \\ &= \|W_\Phi f\|_{A_p^2}^2. \end{aligned}$$

Here A_p^2 is the weighted Bergman space over \mathbb{B} defined by

$$A_p^2 := \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_{A_p^2} = \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |z|^2)^p dV(z) \right)^{1/2} < \infty \right\}.$$

Combing the above formulas, we have

$$(2.2) \quad \|f\|_p = \sup \left\{ \|W_\Phi f\|_{A_p^2} : \Phi \in \text{Aut}(\mathbb{B}) \right\}.$$

It can be checked by a direct calculation that for any two automorphisms Φ and Ψ in $\text{Aut}(\mathbb{B})$, there exists a complex number λ with modulus one such that $W_\Phi W_\Psi = \lambda W_{\Psi \circ \Phi}$. Consequently,

$$\begin{aligned} \|W_\Psi f\|_p &= \sup \left\{ \|W_\Phi W_\Psi f\|_{A_p^2} : \Phi \in \text{Aut}(\mathbb{B}) \right\} \\ &= \sup \left\{ \|W_{\Psi \circ \Phi} f\|_{A_p^2} : \Phi \in \text{Aut}(\mathbb{B}) \right\} = \|f\|_p. \end{aligned}$$

The argument above proves the following result.

Theorem 2.2. *For any automorphism Ψ of the unit ball \mathbb{B} , the weighted composition operator W_Ψ is a surjective isometry on \mathcal{N}_p .*

Recall that a space \mathcal{X} of functions defined on \mathbb{B} is said to be *Moebius-invariant*, or simply *\mathcal{M} -invariant*, if $f \circ \Phi \in \mathcal{X}$ for every $f \in \mathcal{X}$ and every $\Phi \in \text{Aut}(\mathbb{B})$ (see, e.g., [7]). As a corollary to Theorem 2.2, we obtain

Corollary 2.3. *The space \mathcal{N}_p is \mathcal{M} -invariant. Moreover, for any $\Phi \in \text{Aut}(\mathbb{B})$, we have*

$$(2.3) \quad \|C_\Phi\| = \|M_{1/k_a}\| = \left(\frac{1 + |a|}{1 - |a|} \right)^{\frac{n+1}{2}},$$

where $a = \Phi^{-1}(0)$.

Proof. Note that for any automorphism Φ on \mathbb{B} , we have $C_\Phi = M_{1/k_a} \circ W_\Phi$. Since $1/k_a$ is a bounded function, it is a multiplier of \mathcal{N}_p and since W_Φ is a surjective isometry, it follows that $\|C_\Phi\| = \|M_{1/k_a}\|$. The last equality in (2.3) follows from the fact that $\|1/k_a\|_\infty = \left(\frac{1+|a|}{1-|a|}\right)^{\frac{n+1}{2}}$ and Theorem 2.1. \square

2.2. Upper estimate of $\|\cdot\|_p$ for small p . It is immediate from the definition of the norm in \mathcal{N}_p that $\|f\|_p \geq \|f\|_{A_p^2}$ for all $p > 0$. As the last result in this section, we provide an upper estimate for $\|f\|_p$ when $p \leq n$.

Proposition 2.4. *For $0 < p \leq n$, there exists a positive constant $C = C(n, p)$ such that for any $f \in \mathcal{N}_p$ we have*

$$\|f\|_p \leq C \left(\int_{\mathbb{B}} \left(\sup_{|w|=|z|} |f(w)|^2 \right) (1 - |z|^2)^p dV(z) \right)^{1/2}.$$

Proof. For $a \in \mathbb{B}$, integration in polar coordinates ([13, Lemma 1.8]) gives

$$\begin{aligned} & \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ &= \int_{\mathbb{B}} |f(z)|^2 \frac{(1 - |z|^2)^p (1 - |a|^2)^p}{|1 - \langle z, a \rangle|^{2p}} dV(z) \\ &= 2n \int_0^1 r^{2n-1} (1 - |r|^2)^p \left(\int_{\mathbb{S}} |f(r\zeta)|^2 \frac{(1 - |a|^2)^p}{|1 - \langle \zeta, ra \rangle|^{2p}} d\sigma(\zeta) \right) dr \\ &\leq 2n \int_0^1 r^{2n-1} (1 - |r|^2)^p \left(\sup_{\zeta \in \mathbb{S}} |f(r\zeta)|^2 \right) \left(\int_{\mathbb{S}} \frac{(1 - |a|^2)^p}{|1 - \langle \zeta, ra \rangle|^{2p}} d\sigma(\zeta) \right) dr. \end{aligned}$$

Now [13, Theorem 1.12] with $a \in \mathbb{B}$ and $0 < r < 1$ gives

$$\begin{aligned} \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|1 - \langle r\zeta, a \rangle|^{2p}} &= \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|1 - \langle \zeta, ra \rangle|^{2p}} = \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|1 - \langle \zeta, ra \rangle|^{n+(2p-n)}} \\ &\simeq \begin{cases} \text{bounded in } \mathbb{B} & \text{for } 0 < p < \frac{n}{2}, \\ \log \frac{1}{1-r^2|a|^2} \leq \log \frac{1}{1-|a|^2} & \text{for } p = \frac{n}{2}, \\ (1 - r^2|a|^2)^{n-2p} \leq (1 - |a|^2)^{n-2p} & \text{for } \frac{n}{2} < p \leq n. \end{cases} \end{aligned}$$

Thus, for all cases of $0 < p \leq n$, there exists a positive constant C independent of a and r such that

$$\int_{\mathbb{S}} \frac{(1 - |a|^2)^p}{|1 - \langle r\zeta, a \rangle|^{2p}} d\sigma(\zeta) \leq C.$$

It then follows that

$$\begin{aligned} & \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ &\leq C \left(2n \int_0^1 r^{2n-1} (1 - |r|^2)^p \sup_{|w|=r} |f(w)|^2 dr \right) \\ &= C \int_{\mathbb{B}} \left(\sup_{|w|=|z|} |f(w)|^2 \right) (1 - |z|^2)^p dV(z). \end{aligned}$$

Taking supremum over $a \in \mathbb{B}$ gives the required inequality. \square

3. THE CLOSURE OF ALL POLYNOMIALS IN \mathcal{N}_p

It is natural to consider what the closure of all the polynomials is in \mathcal{N}_p -spaces. We introduce the little space \mathcal{N}_p^0 of \mathcal{N}_p , which is defined as

$$\mathcal{N}_p^0 = \mathcal{N}_p^0(\mathbb{B}) = \left\{ f \in \mathcal{N}_p : \lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) = 0 \right\}.$$

In this section, we show that the closure of all the polynomials on \mathbb{B} coincides with the little space \mathcal{N}_p^0 .

Proposition 3.1. *\mathcal{N}_p^0 is a closed subspace of \mathcal{N}_p , and hence it is a Banach space.*

Proof. It can be easily shown that \mathcal{N}_p^0 is a subspace of \mathcal{N}_p and hence it suffices to show that \mathcal{N}_p^0 is closed.

Consider a sequence $\{f_n\} \subset \mathcal{N}_p^0$ that converges to some $f \in \mathcal{N}_p$. We show that $f \in \mathcal{N}_p^0$. Indeed, for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$, such that $\|f - f_n\|_p < \sqrt{\frac{\varepsilon}{4}}$, $\forall n > N$. Let $n_0 > N$ be fixed. Since $f_{n_0} \in \mathcal{N}_p^0$, there exists a $\delta \in (0, 1)$, such that

$$\sup_{\delta < |a| < 1} \int_{\mathbb{B}} |f_{n_0}(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) < \frac{\varepsilon}{4}.$$

As a consequence,

$$\begin{aligned} & \sup_{\delta < |a| < 1} \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ & \leq \sup_{\delta < |a| < 1} \int_{\mathbb{B}} 2(|f(z) - f_{n_0}(z)|^2 + |f_{n_0}(z)|^2) (1 - |\Phi_a(z)|^2)^p dV(z) \\ & \leq 2\|f - f_{n_0}\|_p^2 + 2 \sup_{\delta < |a| < 1} \int_{\mathbb{B}} |f_{n_0}(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) < \varepsilon, \end{aligned}$$

which implies

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) = 0.$$

From this, we conclude that \mathcal{N}_p^0 is closed. \square

Lemma 3.2. *For any $p > 0$, we have $A^2 \subset \mathcal{N}_p^0$.*

Proof. Let f be an element in A^2 . For $a \in \mathbb{B}$, define

$$g_a(z) = |f(z)|^2 (1 - |\Phi_a(z)|^2)^p = |f(z)|^2 \frac{(1 - |a|^2)^p (1 - |z|^2)^p}{|1 - \langle z, a \rangle|^{2p}}, \quad z \in \mathbb{B}.$$

We have $0 \leq g_a(z) \leq |f(z)|^2$ and $\lim_{|a| \rightarrow 1^-} |g_a(z)| = 0$ for all $z \in \mathbb{B}$. The Dominated Convergence Theorem then implies

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) = \lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}} g_a(z) dV(z) = 0.$$

This shows that f belongs to \mathcal{N}_p^0 . \square

Theorem 3.3. *Suppose $f \in \mathcal{N}_p$. Then $f \in \mathcal{N}_p^0$ if and only if*

$$\|f_r - f\|_p \rightarrow 0 \text{ as } r \rightarrow 1^-,$$

where $f_r(z) = f(rz)$ for all $z \in \mathbb{B}$.

Proof. • **Necessity.** Suppose $f \in \mathcal{N}_p^0$. This implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that with $\delta < |a| < 1$, we have

$$(3.1) \quad \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) < \frac{\varepsilon}{6 \cdot 4^n}.$$

Furthermore, by Schwarz-Pick Lemma (see, e.g., [7, Theorem 8.1.4]), we have

$$(3.2) \quad |\Phi_{ra}(rz)| \leq |\Phi_a(z)| \text{ for all } r \in (0, 1) \text{ and } a, z \in \mathbb{B}.$$

Now take and fix $\delta_0 \in (\delta, 1)$. Consider r satisfying $\max \left\{ \frac{1}{2}, \frac{\delta}{\delta_0} \right\} < r < 1$. In this case, for all $a \in \mathbb{B}$ with $|a| \in (\delta_0, 1)$, by (3.1) and (3.2), we have

$$\begin{aligned} & \int_{\mathbb{B}} |f(rz)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ & \leq \int_{\mathbb{B}} |f(rz)|^2 (1 - |\Phi_{ra}(rz)|^2)^p dV(z) \\ & = \left(\frac{1}{r} \right)^{2n} \int_{r\mathbb{B}} |f(w)|^2 (1 - |\Phi_{ra}(w)|^2)^p dV(w) \\ & \leq 4^n \int_{\mathbb{B}} |f(w)|^2 (1 - |\Phi_{ra}(w)|^2)^p dV(w) < \frac{\varepsilon}{6}. \end{aligned}$$

On the other hand, since $f \in A_p^2$, by [13, Proposition 2.6], $f(rz)$ converges to $f(z)$ as $r \rightarrow 1^-$, in the norm topology of the Bergman space $A_p^2(\mathbb{B})$. This implies that there exists a $r_1 \in (0, 1)$ such that for $r_1 < r < 1$, we have

$$\int_{\mathbb{B}} |f(rz) - f(z)|^2 (1 - |z|^2)^p dV(z) < \frac{(1 - \delta_0)^{2p} \cdot \varepsilon}{3}.$$

Consequently, for $|a| \leq \delta_0$ and $r_1 < r < 1$, we have

$$\begin{aligned} & \sup_{|a| \leq \delta_0} \int_{\mathbb{B}} |f(rz) - f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ & = \sup_{|a| \leq \delta_0} \left\{ (1 - |a|^2)^p \int_{\mathbb{B}} |f(rz) - f(z)|^2 \frac{(1 - |z|^2)^p}{|1 - \langle z, a \rangle|^{2p}} dV(z) \right\} \\ & \leq \sup_{|a| \leq \delta_0} \int_{\mathbb{B}} |f(rz) - f(z)|^2 \frac{(1 - |z|^2)^p}{|1 - \langle z, a \rangle|^{2p}} dV(z) \\ & \leq \frac{1}{(1 - \delta_0)^{2p}} \int_{\mathbb{B}} |f(rz) - f(z)|^2 (1 - |z|^2)^p dV(z) < \frac{\varepsilon}{3}. \end{aligned}$$

For all r with $\max \left\{ \frac{1}{2}, \frac{\delta}{\delta_0}, r_1 \right\} < r < 1$, combining the above estimates yields

$$\begin{aligned} \|f_r - f\|_p^2 &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(rz) - f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ &\leq \left(\sup_{|a| \leq \delta_0} + \sup_{\delta_0 < |a| < 1} \right) \int_{\mathbb{B}} |f(rz) - f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ &\leq \frac{\varepsilon}{3} + 2 \sup_{\delta_0 < |a| < 1} \int_{\mathbb{B}} (|f(rz)|^2 + |f(z)|^2) (1 - |\Phi_a(z)|^2)^p dV(z) \\ &< \frac{\varepsilon}{3} + 2 \left(\frac{\varepsilon}{6} + \frac{\varepsilon}{6 \cdot 4^n} \right) < \frac{\varepsilon}{3} + 2 \left(\frac{\varepsilon}{6} + \frac{\varepsilon}{6} \right) = \varepsilon, \end{aligned}$$

which show that $\|f_r - f\|_p \rightarrow 0$ as $r \rightarrow 1^-$.

• **Sufficiency.** Suppose that $\|f_r - f\|_p \rightarrow 0$, as $r \rightarrow 1^-$. For each $0 < r < 1$, the holomorphic function f_r is bounded, hence it belongs to \mathcal{N}_p^0 by Lemma 3.2. Since \mathcal{N}_p^0 is a closed subspace of \mathcal{N}_p , it follows that f belongs to \mathcal{N}_p^0 . \square

As a corollary to Theorem 3.3, we obtain

Corollary 3.4. *The set of polynomials is dense in \mathcal{N}_p^0 .*

Proof. By Theorem 3.3, for any $f \in \mathcal{N}_p^0$, we have

$$\lim_{r \rightarrow 1^-} \|f_r - f\|_p = 0.$$

Since each f_r can be uniformly approximated by polynomials, and moreover, by Theorem 1.1, the sup-norm dominates the \mathcal{N}_p -norm, we conclude that every $f \in \mathcal{N}_p^0$ can be approximated in the \mathcal{N}_p -norm by polynomials. \square

4. \mathcal{N}_p -NORM VIA CARLESON MEASURES

Recall (see, e.g., [13]) that for $\xi \in \mathbb{S}$ and $r > 0$, a Carleson tube at ξ is defined as

$$Q_r(\xi) = \{z \in \mathbb{B} : |1 - \langle z, w \rangle| < r\}.$$

A positive Borel measure μ in \mathbb{B} is called a *p-Carleson measure* if there exists a constant $C > 0$ such that

$$\mu(Q_r(\xi)) \leq Cr^p$$

for all $\xi \in \mathbb{S}$ and $r > 0$. Moreover, if

$$\lim_{r \rightarrow 0} \frac{\mu(Q_r(\xi))}{r^p} = 0$$

uniformly for $\xi \in \mathbb{S}$, then μ is called a *vanishing p-Carleson measure*.

The following result describes a relationship between functions in \mathcal{N}_p as well as \mathcal{N}_p^0 and Carleson measures.

Proposition 4.1. *Let $p > 0$ and $f \in \mathcal{O}(\mathbb{B})$. Define $d\mu_{f,p}(z) = |f(z)|^2(1 - |z|^2)^p dV(z)$. The following assertions hold.*

- (1) *$f \in \mathcal{N}_p$ if and only if $\mu_{f,p}$ is a p-Carleson measure.*
- (2) *$f \in \mathcal{N}_p^0$ if and only if $\mu_{f,p}$ is a vanishing p-Carleson measure.*

Moreover, it holds

$$(4.1) \quad \|f\|_p^2 \simeq \sup_{r \in (0,1), \xi \in \mathbb{S}} \frac{\mu_{f,p}(Q_r(\xi))}{r^p} = \sup_{r \in (0,1), \xi \in \mathbb{S}} \frac{1}{r^p} \int_{Q_r(\xi)} |f(z)|^2(1 - |z|^2)^p dV(z).$$

Proof. For any $f \in \mathcal{O}(\mathbb{B})$, we can write

$$\begin{aligned} \|f\|_p^2 &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^2 \frac{(1 - |a|^2)^p (1 - |z|^2)^p}{|1 - \langle a, z \rangle|^{2p}} dV(z) \\ &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\frac{1 - |a|^2}{|1 - \langle a, z \rangle|^2} \right)^p d\mu_{f,p}(z). \end{aligned}$$

Then statement (1) as well as (4.1) follow from [14, Theorem 45].

On the other hand, statement (2) is a consequence of the “little-oh version” of [14, Theorem 45], which we provide a detailed proof below. \square

Lemma 4.2. *Let $p = n + 1 + \alpha > 0$ and μ be a finite positive Borel measure on \mathbb{B} . Then the following conditions are equivalent.*

- (a) μ is a vanishing p -Carleson measure.
- (b) For each $s > 0$,

$$(4.2) \quad \lim_{|z| \rightarrow 1^-} \int_{\mathbb{B}} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} = 0.$$

- (c) For some $s > 0$, (4.2) holds.

Proof. • The implication (b) \implies (c) is obvious.

• (c) \implies (a): Suppose the condition (c) holds. This means that there exists $s > 0$, such that

$$\lim_{|z| \rightarrow 1^-} \int_{\mathbb{B}} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} = 0.$$

Then for any $\varepsilon > 0$, there exists $\delta > 0$, such that when $\delta < |z| < 1$,

$$(4.3) \quad \int_{\mathbb{B}} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} < \varepsilon.$$

We first show that μ is a p -Carleson measure. Indeed, for $|z| \leq \delta$, we have

$$\int_{\mathbb{B}} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} \leq \int_{\mathbb{B}} \frac{d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} \leq \frac{\mu(\mathbb{B})}{(1 - \delta)^{p+s}} < \infty.$$

This fact and (4.3) show that μ satisfies condition (c) in [14, Theorem 45]. Consequently, μ is a p -Carleson measure.

Next we prove that μ is a vanishing p -Carleson measure. Let ξ be in \mathbb{S} . For $r \in (0, 1 - \delta)$, put $z = (1 - r)\xi$. Then $\delta < |z| < 1$ and for any $w \in \mathbb{Q}_r(\xi)$,

$$|1 - \langle z, w \rangle| = |(1 - r)(1 - \langle \xi, w \rangle) + r| \leq (1 - r)r + r < 2r.$$

Consequently,

$$\frac{(1 - |z|^2)^s}{|1 - \langle z, w \rangle|^{p+s}} \geq \frac{(1 - |z|)^s}{|1 - \langle z, w \rangle|^{p+s}} \geq \frac{r^s}{(2r)^{p+s}} = \frac{2^{-(p+s)}}{r^p}.$$

Using (4.3), we obtain

$$\begin{aligned} \frac{\mu(Q_r(\xi))}{r^p} &= \frac{1}{r^p} \int_{Q_r(\xi)} d\mu(w) \leq 2^{p+s} \int_{Q_r(\xi)} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} \\ &\leq 2^{p+s} \int_{\mathbb{B}} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} < 2^{p+s} \varepsilon, \end{aligned}$$

which implies (a).

• (a) \implies (b): Suppose (a) holds, which means that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 < r < \delta$, we have

$$(4.4) \quad \frac{\mu(Q_r(\xi))}{r^p} < \varepsilon \quad \text{for all } \xi \in \mathbb{S}.$$

Also, since μ is a Carleson measure, there is a positive constant C such that

$$(4.5) \quad \mu(Q_r(\xi)) \leq Cr^p \quad \text{for all } \xi \in \mathbb{S} \text{ and } 0 < r < 1.$$

Let $s > 0$. For the same ε chosen above, take $N_0 \in \mathbb{N}$ such that

$$(4.6) \quad \sum_{k=N_0+1}^{\infty} \frac{1}{2^{sk}} < \varepsilon.$$

Take and fix some $z \in \mathbb{B}$ with $\max\{\frac{3}{4}, 1 - \frac{\delta}{2^{N_0+1}}\} < |z| < 1$ and set $\xi = z/|z|$. For any nonnegative integer k , let $r_k = 2^{k+1}(1 - |z|)$. We decompose the unit ball \mathbb{B} into the disjoint union of the following sets:

$$E_0 = Q_{r_0}(\xi), \quad E_k = Q_{r_k}(\xi) \setminus Q_{r_{k-1}}(\xi), \quad 1 \leq k < \infty.$$

For $k \geq 2$ and $w \in E_k$, we have

$$(4.7) \quad \begin{aligned} |1 - \langle z, w \rangle| &= | |z|(1 - \langle \xi, w \rangle) + (1 - |z|) | \\ &\geq |z| |1 - \langle \xi, w \rangle| - (1 - |z|) \\ &\geq (3/4)2^k(1 - |z|) - (1 - |z|) \\ &\geq 2^{k-1}(1 - |z|). \end{aligned}$$

This also holds for $k = 1$ and $k = 0$, because

$$|1 - \langle z, w \rangle| \geq 1 - |z| \geq \frac{1}{2}(1 - |z|).$$

Now we consider two cases of $k \in \mathbb{N}$.

- Case I: $0 \leq k \leq N_0$. In this case, we have

$$0 < r_k = 2^{k+1}(1 - |z|) \leq 2^{N_0+1}(1 - |z|) < \delta.$$

This implies, by (4.4), that

$$(4.8) \quad \mu(E_k) \leq \mu(Q_{r_k}(\xi)) \leq r_k^p \varepsilon = 2^{p(k+1)}(1 - |z|)^p \varepsilon.$$

- Case II: $k > N_0$. Using (4.5), we have

$$(4.9) \quad \mu(E_k) \leq \mu(Q_{r_k}(\xi)) \leq 2^{p(k+1)}(1 - |z|)^p C.$$

For $\max\{\frac{3}{4}, 1 - \frac{\delta}{2^{N_0+1}}\} < |z| < 1$, using (4.6), (4.7), (4.8) and (4.9), we compute

$$\begin{aligned} \int_{\mathbb{B}} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} &= \sum_{k=0}^{\infty} \int_{E_k} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} \\ &\leq \sum_{k=0}^{\infty} \frac{(1 - |z|^2)^s \mu(E_k)}{(2^{k-1}(1 - |z|))^{p+s}} = \left(\sum_{k=0}^{N_0} + \sum_{k=N_0+1}^{\infty} \right) \frac{(1 - |z|^2)^s \mu(E_k)}{(2^{k-1}(1 - |z|))^{p+s}} \\ &\leq \sum_{k=0}^{N_0} \frac{2^{s+p(k+1)}(1 - |z|)^{p+s} \varepsilon}{2^{(k-1)(p+s)}(1 - |z|)^{p+s}} + \sum_{k=N_0}^{\infty} \frac{2^{s+p(k+1)}(1 - |z|)^{p+s} C}{2^{(k-1)(p+s)}(1 - |z|)^{p+s}} \\ &= \varepsilon \sum_{k=0}^{N_0} \frac{2^{s+p(k+1)}}{2^{(k-1)(p+s)}} + C \sum_{k=N_0+1}^{\infty} \frac{2^{s+p(k+1)}}{2^{(k-1)(p+s)}} \\ &= \varepsilon \cdot 4^{s+p} \sum_{k=0}^{N_0} \frac{1}{2^{ks}} + C \cdot 4^{p+s} \sum_{k=N_0+1}^{\infty} \frac{1}{2^{ks}} \\ &\leq \frac{4^{s+p}}{1 - 2^{-s}} \varepsilon + 4^{p+s} C \varepsilon = M \varepsilon, \end{aligned}$$

where M is a constant depending only on s and p . This shows that statement (b) holds. \square

5. DIFFERENCES OF \mathcal{N}_p FOR SMALL VALUES OF p

In this section, we show that for p small, that is $0 < p \leq n$, all \mathcal{N}_p -spaces are different. This together with Theorem 1.1 (b) gives a complete relationship between \mathcal{N}_p -spaces for all $p > 0$.

We prove this fact by a construction. In [8], the authors constructed a sequence of homogeneous polynomials $(P_k)_{k \in \mathbb{N}}$ satisfying $\deg(P_k) = k$,

$$(5.1) \quad \|P_k\|_\infty = \sup_{\xi \in \mathbb{S}} |P_k(\xi)| = 1, \text{ and } \left(\int_{\mathbb{S}} |P_k(\xi)|^2 d\sigma(\xi) \right)^{1/2} \geq \frac{\sqrt{\pi}}{2^n}.$$

Note that the homogeneity of P_k implies that $|P_k(z)| \leq |z|^k$ for all $z \in \mathbb{B}$.

Let $\{m_k\}_{k=0}^\infty$ be a sequence of positive integers such that $m_{k+1}/m_k \geq c$ for all $k \geq 0$, where $c > 1$ is a constant. Let

$$(5.2) \quad f(z) = \sum_{k=0}^{\infty} b_k P_{m_k}(z) \text{ for } z \in \mathbb{B}.$$

Such a function is said to belong to the *Hadamard gap class*. A characterization for a Hadamard gap class function to be in a weighted Bergman space was given in [10]. In the following result, we obtain an estimate for the \mathcal{N}_p -norm and A^{-q} -norm of f . These results are higher dimensional versions of [6, Theorem 3.3].

Theorem 5.1. *Let f be defined as in (5.2). Let p be a positive real number. Then the following statements hold:*

- (a) *For $0 < p \leq n$, we have $\|f\|_p^2 \simeq \sum_{k=0}^{\infty} \frac{|b_k|^2}{m_k^{p+1}}$.*
- (b) *For any $q > 0$, we have $|f|_q \simeq \sup_k \frac{|b_k|}{m_k^q}$.*

(Here, $\|f\|_p$ and $|f|_q$ denote the norm of f in the spaces \mathcal{N}_p and A^{-q} , respectively).

Note that Theorem 5.1, for $n = 1$, contains the corresponding results in [6] as particular cases.

Proof. (a) Consider $0 < p \leq n$. Since $|P_{m_k}(w)| \leq |w|^{m_k}$ for all $k \geq 0$ and $w \in \mathbb{B}$, we have

$$\sup_{|w|=|z|} |f(w)| \leq \sum_{k=0}^{\infty} |b_k| |z|^{m_k}$$

for any $z \in \mathbb{B}$. Proposition 2.4 and integration in polar coordinates then give

$$\|f\|_p^2 \lesssim \int_0^1 \left(\sum_{k=0}^{\infty} |b_k| r^{m_k} \right)^2 (1-r^2)^p dr.$$

On the other hand, by [5, Theorem 1],

$$\int_0^1 \left(\sum_{k=0}^{\infty} |b_k| r^{m_k} \right)^2 (1-r^2)^p dr \simeq \sum_{k=0}^{\infty} 2^{-k(p+1)} \left(\sum_{2^k \leq m_j < 2^{k+1}} |b_j| \right)^2.$$

Since $m_{j+1} \geq c m_j$ for all j , the cardinality of $\{j : 2^k \leq m_j < 2^{k+1}\}$ is at most $1 + \log_c 2$. It then follows that

$$\begin{aligned} \sum_{k=0}^{\infty} 2^{-k(p+1)} \left(\sum_{2^k \leq m_j < 2^{k+1}} |b_j| \right)^2 &\lesssim \sum_{k=0}^{\infty} 2^{-k(p+1)} \left(\sum_{2^k \leq m_j < 2^{k+1}} |b_j|^2 \right) \\ &\lesssim \sum_{k=0}^{\infty} \left(\sum_{2^k \leq m_j < 2^{k+1}} m_j^{-(p+1)} |b_j|^2 \right) \\ &= \sum_{k=0}^{\infty} \frac{|b_k|^2}{m_k^{p+1}}. \end{aligned}$$

Combining the above estimates, we obtain $\|f\|_p^2 \lesssim \sum_{j=0}^{\infty} \frac{|b_k|^2}{m_k^{p+1}}$.

To prove the reverse inequality, we use the orthogonality of homogeneous polynomials of different degrees in A_p^2 to obtain

$$\begin{aligned} \|f\|_p^2 &\geq \|f\|_{A_p^2}^2 = \int_{\mathbb{B}} \left| \sum_{k=0}^{\infty} b_k P_{m_k}(z) \right|^2 (1 - |z|^2)^p dV(z) \\ &= \sum_{k=0}^{\infty} |b_k|^2 \int_{\mathbb{B}} |P_{m_k}(z)|^2 (1 - |z|^2)^p dv(z) \\ &= \sum_{k=0}^{\infty} |b_k|^2 \int_0^1 2n r^{2n+2m_k-1} (1 - r^2)^p dr \int_{\mathbb{S}} |P_{m_k}(\xi)|^2 d\sigma(\xi) \\ &\gtrsim \sum_{k=0}^{\infty} |b_k|^2 \int_0^1 t^{n+m_k-1} (1 - t)^p dt \\ &\quad (\text{by (5.1) and the change of variables } t = r^2) \\ &= \sum_{k=0}^{\infty} |b_k|^2 \frac{\Gamma(n + m_k) \Gamma(p + 1)}{\Gamma(n + m_k + p + 1)} \gtrsim \sum_{k=0}^{\infty} \frac{|b_k|^2}{m_k^{p+1}}. \end{aligned}$$

The last inequality follows from Stirling's formula. We have thus completed the proof of (a).

(b) Assume $f \in A^{-q}$ for $q > 0$. Fix a positive integer k . For $r > 0$ and $\xi \in \mathbb{S}$, we have $|f|_q^2 (1 - r)^{-2q} \geq |f(r\xi)|^2$. Integrating with respect to $\xi \in \mathbb{S}$ and using (5.1) yield

$$\begin{aligned} \frac{|f|_q^2}{(1 - r)^{2q}} &\geq \int_{\mathbb{S}} |f(r\xi)|^2 d\sigma(\xi) = \int_{\mathbb{S}} \left| \sum_{j=0}^{\infty} b_j r^{m_j} P_{m_j}(\xi) \right|^2 d\sigma(\xi) \\ &= \sum_{j=0}^{\infty} |b_j|^2 r^{2m_j} \int_{\mathbb{S}} |P_{m_j}(\xi)|^2 d\sigma(\xi) \gtrsim |b_k|^2 r^{2m_k}. \end{aligned}$$

Setting $r = m_k/(q + m_k)$, we obtain

$$|b_k| \lesssim \frac{|f|_q}{r^{m_k} (1 - r)^q} = |f|_q \left(1 + \frac{q}{m_k} \right)^{m_k} \left(1 + \frac{m_k}{q} \right)^q \lesssim |f|_q m_k^q,$$

which implies $\sup_k \frac{|b_k|}{m_k^q} \lesssim |f|_q$.

Put $L = \sup_k \{|b_k| m_k^{-q}\}$ so $|b_k| \leq L m_k^q$ for all $k \geq 0$. For each $z \in \mathbb{B}$, we have

$$\begin{aligned}
 \frac{|f(z)|}{1-|z|} &\leq \left(\sum_{k=0}^{\infty} |b_k| |P_{m_k}(z)| \right) \left(\sum_{s=0}^{\infty} |z|^s \right) \\
 (5.3) \quad &\leq L \left(\sum_{k=0}^{\infty} m_k^q |z|^{m_k} \right) \left(\sum_{s=0}^{\infty} |z|^s \right) \\
 &\quad (\text{since } |P_{m_k}(z)| \leq |z|^{m_k} \text{ for all } k \geq 0) \\
 &= L \sum_{\ell=1}^{\infty} \left(\sum_{m_k \leq \ell} m_k^q \right) |z|^\ell.
 \end{aligned}$$

Since $m_{k+1}/m_k \geq c > 1$ for all k , we have

$$\sum_{m_k \leq \ell} m_k^q = \ell^q \sum_{m_k \leq \ell} \left(\frac{m_k}{\ell} \right)^q \leq \ell^q \sum_{s=0}^{\infty} (c^{-q})^s = \frac{\ell^q}{1-c^{-q}}.$$

By Stirling's formula, it follows that

$$(5.4) \quad \sum_{m_k \leq \ell} m_k^q \leq \frac{\ell^q}{1-c^{-q}} \lesssim \frac{1}{1-c^{-q}} \frac{\Gamma(\ell+q+1)}{\Gamma(\ell+1)\Gamma(q)}.$$

Combing (5.3) and (5.4) yields

$$\frac{|f(z)|}{1-|z|} \lesssim \frac{L}{1-c^{-q}} \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell+q+1)}{\Gamma(\ell+1)\Gamma(q)} |z|^\ell = \frac{L}{1-c^{-q}} \frac{1}{(1-|z|)^{q+1}}.$$

Consequently,

$$|f|_q = \sup_{z \in \mathbb{B}} |f(z)| (1-|z|)^q \lesssim L = \sup_k \frac{|b_k|}{m_k^q}.$$

This completes to proof of (b). \square

Corollary 5.2. *If $0 < p_1 < p_2 \leq n$, then we have*

$$\mathcal{N}_{p_1} \subsetneq \mathcal{N}_{p_2} \subsetneq A^{-\frac{n+1}{2}}.$$

Proof. Define

$$(5.5) \quad f_1(z) = \sum_{k=0}^{\infty} 2^{\frac{k(n+1)}{2}} P_{2^k}(z), \quad f_2(z) = \sum_{k=0}^{\infty} 2^{\frac{k(1+p_1)}{2}} P_{2^k}(z)$$

for $z \in \mathbb{B}$. Using Theorem 5.1, it can be checked with a direct computation that $f_1 \in A^{-\frac{n+1}{2}} \setminus \mathcal{N}_{p_2}$ and $f_2 \in \mathcal{N}_{p_2} \setminus \mathcal{N}_{p_1}$. \square

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